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Dynamic instability of a beam undergoing periodic motions over supports

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Abstract

This paper investigates the dynamic stability of a beam moving over two bi-lateral supports using finite element analysis, with essential conditions applied via the penalty function approach. Computational advantages of the penalty-based approach compared to that of the Lagrangian multipliers are highlighted in the context of such unique problems in dynamics. Penalty-based numerical formulation for the moving beam results in a system of second-order differential equations with periodic coefficients. The governing equations are reduced to state-space form and Floquet–Lyapunov theory is applied to investigate dynamic stability of the moving beam. The instability characteristics are studied for a range of amplitudes and frequencies for sinusoidal longitudinal motions of the beam. In addition to the predictions using Floquet–Lyapunov theory, further instability regions based on first and higher approximations are identified. The instability results for periodic motion compare well with previous research and new results are presented taking into consideration the effect of damping. The penalty-based finite element formulation is found to be effective when applied to this class of dynamics problems. The avenues for further research are also highlighted.

1. Introduction

Linear systems with constant parameters exhibit resonance whenever an excitation frequency is same as the natural frequency. In such ordinary resonance cases, the amplitudes of a dynamic system increase linearly or follow the power law in most cases. Systems that are dependent on time are described by differential equations with periodic coefficients, i.e. the system properties such as stiffness, inertia and damping tend to change with time. Such time-dependent systems are characterized by parametric resonances that occur due to a combination of parameters that lead to instability. Generally, parametric instabilities are centered about a union of several smaller regions of frequencies. These aspects render parametric resonance as "more dangerous" or "useful", compared to ordinary resonance and hence prediction of instability regions is of immense importance in engineering applications [1–5].

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Nomenclature

- a_b^L longitudinal acceleration of the beam, m/sec^2
- $a_0, a_1 \ldots a_b$, damping constants
- A amplitude of periodic motion, m
- $[\mathbf{C}\ (\mathbf{t})]$ global damping matrix
- D distance between the supports, m
- E elastic modulus, N/m^2
- $\{\mathbf{F}(\mathbf{t})\}$, $\{\mathbf{P}\ (\mathbf{t})\}$, $\{\mathbf{R}\ (\mathbf{t})\}$, $\{\mathbf{R}^*\ (\mathbf{t})\}$ load vectors
- F_x axial force, N
- I moment of inertia, m^4
- $[\mathbf{K}(\mathbf{t})]$ global stiffness matrix
- $[\mathbf{K}_a]$ time dependent geometric stiffness matrix due to axial periodic force
- $[\mathbf{K}_a]$, $[\mathbf{G}]$, $[\mathbf{K}_\lambda]$ constraint matrices
- $[\mathbf{K}_f]$ flexural stiffness matrix
- l_j length of an element
- L length of the beam, m
- $[\mathbf{M}]$ global mass matrix
- $[\mathbf{P}_{\text{flo}}]$ floquet matrix
- $[\mathbf{P}_T]$ state transition matrix
- $\{q\}$ generalized coordinates
- T time period, sec
- T time period
- α, λ penalty and Lagrangian multiplier constraints
- α_i weights for the Gaussian quadrature
- α_p penalty multiplier
- $\bar{\alpha}_p$ real part
- γ beam mass per unit length, kg m
- γ mass per unit length, N/m
- λ_c combination resonance frequency, rad/s
- λ_i, λ_j natural frequencies, rad/s
- λ_{real}, λ_{img} real and imaginary parts of Λ
- ω_p imaginary part
- ω longitudinal oscillation frequency, rad/s
- ξ Gauss–Legendre points
- ψ damping ratio
- Λ eigenvalues of the transition matrix
- $\bar{\Pi}$ total potential

Systems with periodic coefficients are seen in many engineering applications such as aero-elasticity and structural dynamics [6,7,12]. There are several techniques that could be used to solve the system of equations with periodic coefficients. The most common techniques are Hill's method [8], perturbation techniques [9], harmonic balance [6], multi-blade coordinates [10] and Floquet–Lyapunov theory [3,4,11]. Explanation on Floquet–Lyapunov theory and related topics on solutions of differential equations with periodic coefficients can be found in Refs. [3,4]. The two key aspects of the theory are the existence and uniqueness of solution to linear differential equations with periodic coefficients and stability of the obtained solution [33–40].

The stability of rotor dynamic and aero-elastic problems using Floquet–Lyapunov theory has been a subject of extensive research for example [2,12]. There have been several attempts in the past that studied axially moving beams in rotation [13–15]. Such classes of problems fall in an altogether different category since one end boundary condition remains restrained in all degrees of freedom.

Perhaps less explored is the instability problem of a flexible beam moving over two bilateral supports, which has not been covered extensively in previous research. Only a few known attempts have studied the dynamics of a finite flexible beam moving over two bi-lateral supports [1,16–18]. The significance of moving beam problems has been highlighted in the literature in association with design of robot manipulators, band saw blades and computer tapes, to mention a few [19–21], Tan et al. (1993). The emphases in previous research have always been on closed form solutions and the use of alternate techniques such as assumed modes that are complex in terms of formulation [1,16,22,23]. Buffinton and Kane [1] as well as Lee [16] used the assumed modes approach for investigating the moving beam response. One of the main limitations in using assumed modes technique is to effectively incorporate change in cross-section area as in stepped beams [1,24] studied the dynamic stability of moving beams using assumed modes method. However, their technique did not include damping effects.

It should be noted that Lee [16] also presented some results on stability of axially moving beams and unsuccessfully tried to compare the results with that of Buffinton and Kane [1]. Previous publications [18] details these grossly incorrect interpretations made by Lee. Thus, Lee's interpretations on dynamic stability for the longitudinally moving beam are not considered for comparison in the present study.

The nature of the essential conditions renders this problem challenging to be approached using classical finite element method. As the location of the supports changes with time, the finite element-based model should be capable of enforcing the support conditions. Using a classical finite element approach, the incorporation of these time-dependent conditions is achieved by having a node at these support locations. Since the basic vibration problem of a moving beam is a space–time problem, the space domain can be divided into finite elements and the governing equations are derived. Then direct time marching is the standard procedure for solving in the time domain [25]. Zienkiewicz and Taylor [27] brought out the limitation of this approach with the requirement of large time steps and increased computational storage. Deriving element shape functions to accommodate such time-dependent effects is a laborious task, if not impossible. This is especially true when the number of supports is large.

Sreeram [17] and Sreeram and Sivaneri [18] were the first to address the response problem [excluding stability] using hp-finite element approach with essential conditions applied via the Lagrangian multipliers. One of the key aspects in Ref. [18] is that the displacement boundary conditions are not forced by having nodes at support locations, rather, by means of Lagrangian multipliers. However, the main limitation associated with the Lagrangian multiplier approach is the ill-conditioning of the system matrices which tend to be positive semi-definite.

For problems requiring the evaluation of dynamic stability, the method of Floquet–Lyapunov is particularly suited and valid for large amplitudes [1,6]. This method requires the reduction of second-order differential equations to the first-order or the state-space form. Due to the presence of zeros, the Lagrangian multiplier method presents computational difficulties in reduction to the desired state-space form. The choice for penalty method over the Lagrangian multiplier method is that the former approach does not introduce additional unknown variables and order of the system equations is not altered. Using the latter approach, numerical difficulties arising due to presence of zero diagonal and off-diagonal elements in mass, damping and stiffness matrices could be avoided. Besides, using the penalty formulation the system matrices are positive definite subject to appropriate choice for penalty numbers [26,27].

Even though a few authors mentioned previously studied the dynamics of axially moving beams, no known attempts have investigated the dynamic stability in detail using finite element approach. There is a need for a finite element-based model to address such unique problems in dynamics from the perspective of both response and stability, and is addressed in this paper. A penalty-based formulation is presented here for the moving beam problem. Also, the formulation includes some form of artificial damping in the model unlike approaches presented in Refs. [1,16,18]. In particular, the effects of distance between the supports, amplitude variation and damping on the dynamic stability characteristics are investigated.

2. Moving beam problem

The basic problem of an axially moving beam [17] is considered here. Fig. 1(a) shows an Euler–Bernoulli beam with overhang resting on two supports. Frame of reference X– Y represents the inertial frame such that the support P is at the origin. The beam bb' of length L moves in the X-direction relative to supports. The axial motion of the beam is described using $X_F\,(t)$ in Fig. 1(a).

The finite element formulation representing the beam dynamics, however, is presented based on the moving frame (x, y) with the inclusion of axial inertia effects. The axial force F_x in the current problem causes the longitudinal rigid body acceleration of the beam and is applied at the left end (marked by *) as shown in Fig. 1(a). This axial forcing function is same as adopted by Lee and Sreeram [16,17]. The beam is divided into several p-elements (Fig. 1(b)) and the shape functions were derived using the Legendre polynomials and is given in detail in Ref. [18]. The axial motion of the beam is described using the function $X_F\,(t)$ and its derivatives. The finite element formulation, however, is presented based on the moving frame (x, y) with the inclusion of axial inertia effects. The inertial force plays a significant role in altering the stiffness matrix as the beam moves periodically. The inertial force distribution is maximum at the left end of the beam and zero at the right end [16,18].

$$F_x = -\int_x^L \gamma a_b^L \, \mathrm{d}x, \tag{1}$$

$$F_x = -\gamma a_b^L (L - x). \tag{2}$$

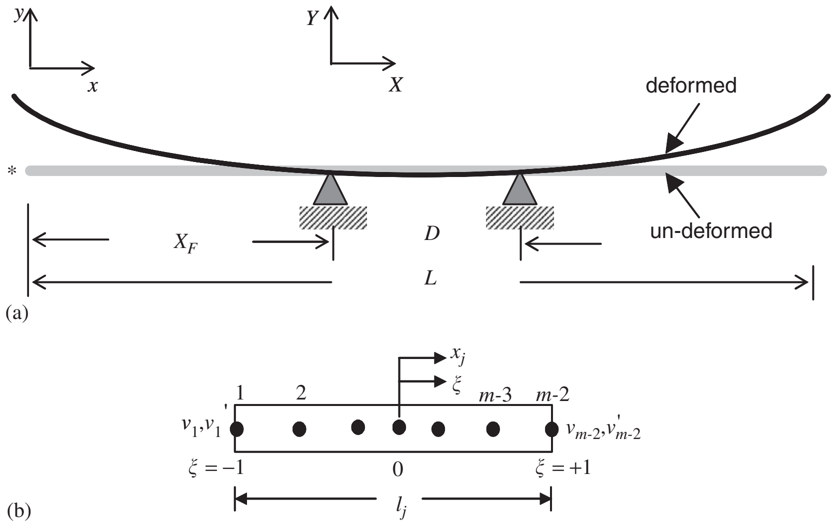

Fig. 1. Moving beam: (a) overhung beam with time-dependent boundary conditions and (b) beam finite element.

3. Finite element formulation

The dynamical equations of motion for the beam include flexural stiffness due to bending matrix and time dependent geometric stiffness matrix due to axial periodic force. A brief overview of the finite element formulation is presented next.

3.1. Beam finite element

In this section, the approach adopted for deriving shape functions for a *p*-version finite element is summarized [17]. Choosing orthogonal polynomials as the basis for building shape functions is convenient during numerical integration wherein the orthogonality property is exploited. Several previous publications, most notably, Hodges (1983), have recommended the use of orthogonal polynomials.

The beam in Fig. 1(b) is divided into a number of *p*-elements. The *j*th element of length l_j is magnified in figure to describe its internal structure. The *j*th element consists of $(m-2)$ nodes numbered $\{1, \ldots, m-2\}$. The zeros of Legendre polynomial coincide with the location of internal nodes. The *j*th element has m degrees of freedom as the end nodes $\{1, m-2\}$ have rotational degrees of freedom. A local coordinate x_j and a non-dimensional coordinate ξ are located at the center of the element as shown in figure; this is assumed to vary from $-l_j/2$ to $l_j/2$ and ξ varies from -1 to $+1$. The transformation between x_j and ξ is given as

$$\xi = \frac{2x_j}{l_j}. \tag{3}$$

The deflection $v(\xi)$ over the element is described by

$$v(\xi) = \sum_{i=0}^{m-1} a_i P_i(\xi), \tag{4}$$

$P_i\ (\xi)$ are the Legendre polynomials of order i and a_i are the undetermined generalized coordinates. In the matrix form

$$v(\xi) = \lfloor P_i(\xi) \rfloor \{a_i\}. \tag{5}$$

The solution of a_i needs m equations. Combining the displacement and slope degrees of freedom in matrix form

$$\begin{bmatrix} P_0(-1) & P_1(-1) & \cdots & P_{m-1}(-1) \\ P'_0(-1) & P'_1(-1) & \cdots & P'_{m-1}(-1) \\ P_0(\xi_2) & P_1(\xi_2) & \cdots & P_{m-1}(\xi_2) \\ \vdots & \vdots & \cdots & \vdots \\ P_0(+1) & P_1(+1) & & P_{m-1}(+1) \\ P'_0(+1) & P'_1(+1) & & P'_{m-1}(+1) \end{bmatrix} \begin{Bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ \vdots \\ a_{m-1} \end{Bmatrix} = \begin{Bmatrix} v_1 \\ \frac{2}{l_j} v'_1 \\ v_2 \\ \vdots \\ \vdots \\ \frac{2}{l_j} v'_{m-1} \end{Bmatrix} \tag{6}$$

or

$$[L]\{a\} = \{q_e\}. \tag{7}$$

The generalized displacements can be expressed as

$$v(\xi) = [P(\xi)][L]^{-1}\{q_e\}. \tag{8}$$

The shape functions are given by

$$H_i(\xi) = [P_0(\xi)P_1(\xi)\cdots P_{m-1}(\xi)][L_i]^{-1}. \tag{9}$$

The shape functions in Eq. (9) are used to generate the system matrices and are discussed next.

3.2. Formation of element matrices

The elastic strain energy due to bending over jth element in terms of the local coordinate is given as

$$\left[K_f^e\right] = \int_{-l_j/2}^{l_j/2} EI\lfloor H''\rfloor\{H''\}\,\mathrm{d}x_j. \tag{10}$$

The strain energy due to the periodic axial force (Eq. (2)) on the element is given as

$$\left[K_a^e\right] = -a_B^L\int_{-l_j/2}^{l_j/2}\left[\gamma(L-x)\lfloor\delta q\rfloor\{H'\}\lfloor H'\rfloor\right]\mathrm{d}x_j. \tag{11}$$

From the kinetic energy of the moving beam, the mass matrix for the jth element is given as

$$[M^e] = \int_{-l_j/2}^{l_j/2}\gamma\lfloor\delta q\rfloor\{H\}\lfloor H\rfloor\{q\}\,\mathrm{d}x_j, \tag{12}$$

or, in terms of the Gauss–Legendre quadrature the bending stiffness, time dependent geometric stiffness matrix and mass matrices are given as

$$\left[K_f^e\right] = \frac{8}{(l_j)^3}\sum_{i=0}^{N_g}\alpha_i EI\lfloor H''(\xi)\rfloor\{H''(\xi)\}, \tag{13}$$

$$\left[K_a^e\right] = \frac{-2a_B^L}{l_j}\left[\sum_{i=0}^{N_g}\alpha_i\gamma\left\{L-\left[x_b^i+\frac{l_j^i}{2}(1+\xi)\right]\right\}\{H'(\xi)\}\lfloor H'(\xi)\rfloor\right], \tag{14}$$

$$[M^e] = \frac{l_j}{2}\sum_{i=0}^{N_g}\alpha_i\gamma\lfloor H(\xi)\rfloor\{H(\xi)\}. \tag{15}$$

The type of problem dictates whether or not damping matrices for a given system can be formed from energy considerations (e.g. damping due to aerodynamic effects). A procedure developed by Banks and Inman [41] considers several types of damping effects such as air damping, Kelvin–Vogit damping, time and spatial hysteresis. Appropriately, the damping constants will also need to be evaluated by experiments.

Such a comprehensive damping model is beyond the scope of this paper. Instead an approximation based on Caughey series is used. From Wilson and Penzien [42], the Caughey series for the formation of damping matrix is given as

$$[C^e] = [M^e]\sum_{b=0}^{N-1} a_b\left[[M^e]^{-1}\left([K_f^e]+[K_a^e]\right)\right]^b. \tag{16}$$

Assuming Rayleigh damping

$$[\mathbf{C^e}] = a_0[M^e] + a_1\left([K_f^e]+[K_a^e]\right). \tag{17}$$

The constants a_0 and a_1 are evaluated for an assumed damping ratio as

$$\{\psi_r\} = \left(\frac{a_0}{\omega_r} + a_1\omega_r\right); \quad r = 1,2\ldots n, \tag{18}$$

or

$$\begin{Bmatrix} \psi_1 \\ \psi_2 \end{Bmatrix} = \frac{1}{2}\begin{bmatrix} \frac{1}{\omega_1} & \omega_1 \\ \frac{1}{\omega_2} & \omega_2 \end{bmatrix}\begin{Bmatrix} a_0 \\ a_1 \end{Bmatrix}. \tag{19}$$

As the beam moves relative to supports, the essential conditions are that at any given instant the displacements at the location of supports are zero. Sreeram and Sivaneri [18] used Lagrangian multipliers for satisfying these essential conditions. The main limitation of Lagrangian multipliers are the introduction of additional unknowns and ill-conditioning of system equations and are discussed in the next section.

4. Mixed formulation for the moving beam

Generally, the boundary conditions are of Dirichlet type or essential boundary value problems (e.g. $u_x = 0$) or Neumann type such as $\partial F/\partial u_y = 0$ specifies a natural boundary condition [26,28]. For the satisfaction of these boundary conditions, two approaches based on mixed formulation are presented here, i.e. the Lagrangian multiplier and penalty functions. A comparison is drawn between the two approaches highlighting the computational advantages in choosing the penalty function-based approach for the stability of the moving beam problem. The computational advantages can be readily seen by considering two cases of time-dependent conditions are considered, (a) a fixed–free beam moving over supports and (b) a free–free beam moving over supports (Fig. 2).

4.1. Lagrangian multiplier method

The Lagrange's method of undetermined multipliers searches for a saddle point by means of minimizing the total potential [26]. Generally, in most cases the system matrices are non-singular and positive definite.

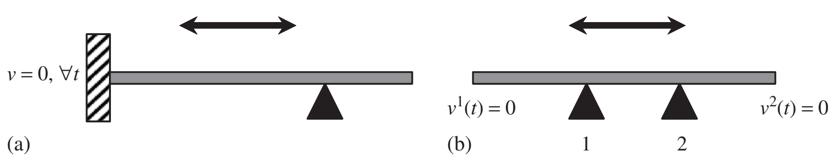

Fig. 2. Moving beam with fixed and free boundary conditions.

For a general problem in dynamics including damping effects depicting cases (a) and (b) can be expressed as

$$\begin{bmatrix} M & 0 \\ 0 & 0 \end{bmatrix}\begin{Bmatrix} \ddot{q} \\ \lambda \end{Bmatrix} + \begin{bmatrix} C(t) & 0 \\ 0 & 0 \end{bmatrix}\begin{Bmatrix} \dot{q} \\ \lambda \end{Bmatrix} + \begin{bmatrix} K(t) & K_\lambda \\ K_\lambda^{\mathrm{T}} & 0 \end{bmatrix}\begin{Bmatrix} q \\ \lambda \end{Bmatrix} = \begin{Bmatrix} \mathbf{R(t)} \\ \mathbf{P(t)} \end{Bmatrix} \tag{20}$$

or in uncoupled form

$$[M]\{\ddot{q}\} + [C(t)]\{\dot{q}\} + [K(t)]\{q\} + [K_\lambda]\{\lambda\} = \{\mathbf{R(t)}\}, \tag{21}$$

$$\left[K_\lambda^{\mathrm{T}}\right]\{q\} = \{\mathbf{P(t)}\}. \tag{22}$$

From Eq. (22) $\{q\}$ is obtained as

$$\{q\} = [K(t)]^{-1}\left[\{\mathbf{R(t)}\} - [M]\{\ddot{q}\} - [C(t)]\{\dot{q}\} - [K_\lambda]\{\lambda\}\right] \tag{23}$$

or

$$\{\lambda\} = \left[K_\lambda^{\mathrm{T}} K(t)^{-1} K_\lambda\right]^{-1}\left[K_\lambda^{\mathrm{T}} K(t)^{-1}(\{\mathbf{R(t)}\} - [M]\{\ddot{q}\} - [C(t)]\{\dot{q}\} - \{\mathbf{R(t)}\}) - \{\mathbf{P(t)}\}\right] \tag{24}$$

or

$$\begin{aligned}[M]\{\ddot{q}\} &+ [C(t)]\{\dot{q}\} + [K(t)]\{q\} \\ &+ \left[K_\lambda^{\mathrm{T}} K(t)^{-1} K_\lambda\right]^{-1}\left[K_\lambda^{\mathrm{T}} K(t)^{-1}(\{\mathbf{R(t)}\} - [C(t)]\{\dot{q}\} - [M]\{\ddot{q}\}) - \{\mathbf{P(t)}\}\right] = \{\mathbf{R(t)}\}\end{aligned} \tag{25}$$

$$\begin{aligned}&\underbrace{[M]\left[I - K_\lambda\left(K_\lambda^{\mathrm{T}} K(t)^{-1} K_\lambda\right)^{-1} K_\lambda^{\mathrm{T}} K(t)^{-1}\right]}_{[M]^*}\{\ddot{q}\} \\ &\quad + \underbrace{[C(t)]\left[I - K_\lambda\left(K_\lambda^{\mathrm{T}} K(t)^{-1} K_\lambda\right)^{-1} K_\lambda^{\mathrm{T}} K(t)^{-1}\right]}_{[C(t)]^*}\{\dot{q}\} + [K(t)]\{q\} \\ &\quad = \underbrace{\{\mathbf{R(t)}\} + K_\lambda\left(K_\lambda^{\mathrm{T}} K(t)^{-1} K_\lambda\right)^{-1}\{\mathbf{P(t)}\} - K_\lambda\left(K_\lambda^{\mathrm{T}} K(t)^{-1} K_\lambda\right)^{-1} K_\lambda^{\mathrm{T}} K(t)^{-1}\{\mathbf{R(t)}\}}_{\{\mathbf{R(t)}\}^*}\end{aligned} \tag{26}$$

or

$$[M]^*\{\ddot{q}\} + [C(t)]^*\{\dot{q}\} + [K(t)]\{q\} = \{\mathbf{R(t)}\}^*. \tag{27}$$

Eq. (27) describes the periodic motion of the beam using the Lagrangian multiplier method for the fixed–free configuration in Fig. 2(a). Eq. (27) is obtained by eliminating the Lagrangian multipliers from the original system Eq. (20). In general, $[\mathbf{K(t)}]$ is positive definite and $[\mathbf{K(t)}]^{-1}$ exists for all types of periodic motion. However, the existence of $[\mathbf{K(t)}]^{-1}$ cannot be ascertained for the free-free beam moving over supports (Fig. 2(b)) since all essential conditions are time dependent, implying singularity associated with unconstrained $[\mathbf{K(t)}]$ in Eq. (20). Moreover, reduction to preferred state-space form beginning from Eq. (20) is computationally intensive, and may lead to ill-conditioning of system equations [29].

4.2. Penalty function method

With the aim of overcoming the limitations of Lagrange multiplier approach, penalty-based formulation is summarized here which at the end does not result in ill-conditioned system equations. In the penalty method, the displacement constraints (essential conditions) are imposed by means of penalty parameters, which are a set of pre-determined constants (penalty numbers). The value of the penalty number (α_p) determines the accuracy of the obtained solution. Hence, in using the penalty-based technique, a penalty number of relatively large magnitude $\alpha_p \gg \max[\mathbf{K(t)_{ii}}]$ is added to the ith diagonal element of $[\mathbf{K(t)}]$. However, the matrices could be ill-conditioned if the off-diagonal terms are multiplied by a large number. One of the important considerations is the appropriate choice for the penalty number and this is arrived at by trials of various values for α_p and checking for convergence of solutions. The penalty method is quite effective since this does not need additional equations thereby preserving the bandwidth of stiffness matrix [**K**(**t**)]. The next section presents the penalty-based technique to enforce essential conditions for the moving beam problem.

5. Penalty-based equations of motion

The equations of motion are derived from the variational expression to which the penalty parameter is added. More generally, the penalty function is applied to minimize the penalized functional as

$$\overline{\prod}_p = \prod_p + \alpha_p \int G^{\mathrm{T}} G \, \mathrm{d}\Omega. \tag{28}$$

The matrix [**G**] contains the constraint equations in domain Ω. α_p is the penalty multiplier and as $\alpha_p \to \infty$ the constraints are satisfied. The elegance of the penalty approach is that the constraints imposed in this way do not introduce additional unknown variables as compared with the Lagrangian multiplier approach presented in Ref. [18]. On the basis of the penalty approach, the finite element dynamic equations of motion is given as

$$[M]\{\ddot{q}\} + [C(t)]\{\dot{q}\} + \left[K_f + K_a + \alpha_p\left(K_\alpha^{\mathrm{T}} K_\alpha\right)\right]\{q\} = \{F(\mathbf{t})\}, \tag{29}$$

$$[M]\{\ddot{q}\} + [C(t)]\{\dot{q}\} + \left[\overline{K}(t)\right]\{q\} = \{F(\mathbf{t})\}. \tag{30}$$

Additionally, as in Eq. (29), the combined stiffness matrix is strongly positive definite in most cases [27]. This may not be the case with the Lagrangian multiplier approach where the stiffness matrices are usually positive semi-definite. However, it should be noted that with exceedingly large values for α_p, the equation may degenerate and could well run into numerical difficulties. There are several approaches to arrive at α_p ranging from trail and error [27] to hybrid methods by Pantano and Averill [30].

Eq. (30) describes the dynamics of moving beam problem including Rayleigh damping effects[1] as given in Eqs. (17)–(19). As can readily be seen from Eq. (30), the moving beam problem formulated using penalty function approach introduces no additional unknown variables. Also, for the present problem, the matrix [**K**(**t**)] in Eq. (30) is non-singular, positive-definite. The constraint matrix [$\mathbf{K}_\alpha$] is evaluated based on the location of the supports. For the case of a moving beam with m degrees of freedom and two supports, the constraint matrix [$\mathbf{K}_\alpha^{\mathrm{T}}$] or [$\mathbf{G}^{\mathrm{T}}$] in Eq. (11) is $2 \times m$ and is given as

$$\left[K_\alpha^{\mathrm{T}}\right] = \begin{bmatrix} 0 & \cdots & H_1^i & H_2^i & \cdots H_m^i & 0 \\ H_1^j & H_2^j & H_m^j \cdots & 0 & 0 & 0 \end{bmatrix}. \tag{31}$$

The constraint matrix in Eq. (31) is formed by evaluating the shape functions H_m^i and H_m^j, evaluated at the ith and jth elements, respectively. The Eq. (31) is not a constant matrix and depending on the location of supports, the matrix is updated for every time step.

6. Stability formulation

Clearly, Eq. (30) is a second-order differential equation with periodic coefficients. The stability conditions are arrived on the basis of the Floquet theory and the Eq. (30) is represented in the first-order or state-space form. Therefore rearranging Eq. (30),

$$\begin{bmatrix} M & 0 \\ 0 & M \end{bmatrix}\begin{Bmatrix} \dot{q} \\ \ddot{q} \end{Bmatrix} - \begin{bmatrix} 0 & M \\ -\overline{K}(t) & -\overline{C}(t) \end{bmatrix}\begin{Bmatrix} q \\ \dot{q} \end{Bmatrix} = \begin{Bmatrix} 0 \\ F(\mathbf{t}) \end{Bmatrix}, \tag{32}$$

$$\{\dot{y}\} - [P_{\mathrm{flo}}(t)]\{y\} = \overline{F}(\mathbf{t}). \tag{33}$$

[1]In this case 5% damping is assumed in the first two modes of vibration. The first two natural frequencies of the beam with the supports at initial position (0.375, 0.625 m) are used to evaluate the damping matrix. The values of a_0 and a_1 are 1.5791 and 0.00405, respectively.

6.1. Formulation based on the Floquet–Lyapunov theorem

The stability analysis based on Floquet theory follows that presented in Refs. [3,4] and only the essential steps are shown here. $[P_{\text{flo}}] \in \Re^n$ is a periodic matrix of order $2N \times 2N$ of period T. All the $2N$ degrees of freedom are required to completely specify the state of the entire system. To determine stability, the load vector $\mathbf{F(t)}$ is set to zero. Floquet theorem states that the solution of Eq. (33) is expressed as

$$Y(t) = X(t)\mathrm{e}^{\lambda_p T}, \tag{34}$$

where $\mathbf{Y(t)}$ and $\mathbf{X(t)}$ are $2N \times 1$ column matrices. By selecting N independent conditions such that $\mathbf{Y(0)} = \mathbf{I}$ solve Eq. (33) over a period T and assemble the $2N \times 2N$ state transition matrix $\mathbf{P}_T$ as

$$[P(t)] = \begin{bmatrix} y^{11} & y^{12} & \cdots & \cdots & \cdots & \cdots & y^{1n} \\ y^{21} & y^{22} & \cdots & & \cdots & \cdots & \cdots \\ y^{31} & y^{32} & \cdots & & \cdots & \cdots & \cdots \\ \cdots & & & \cdots & & \cdots & \cdots \\ \cdots & \cdots & & & \cdots & & \cdots \\ \cdots & \cdots & & & & \cdots & \cdots \\ y^{n1} & y^{n2} & \cdots & \cdots & \cdots & \cdots & y^{nn} \end{bmatrix}. \tag{35}$$

Each column of $[\mathbf{P}_T]$, i.e. $(y^{11}, y^{21} \cdots y^{n1})$ represents a solution obtained by integrating Eq. (33) between (0, T) for a specific initial condition. Evaluating the eigenvalues of the transition matrix

$$\Lambda = \mathrm{e}^{\lambda_p T}, \tag{36}$$

$$\mathrm{e}^{\lambda_p T} = \frac{1}{T}\ln(\lambda), \tag{37}$$

$$\overline{\alpha}_p = \frac{1}{2T}\ln\left(\lambda^2_{\text{real}} + \lambda^2_{\text{img}}\right), \tag{38}$$

$$\omega_p = \frac{1}{T}\tan^{-1}\left[\frac{\lambda_{\text{img}}}{\lambda_{\text{real}}}\right]. \tag{39}$$

The real part λ_{real} is the growth of decay implying that any value greater than unity indicates instability. The system defined by Eq. (33) or alternately, Eq. (30) is stable if and only if the spectral radius of the transition matrix is less than unity. In the case of ordinary resonance, the spectrum of frequencies with indefinitely increasing amplitudes (A) may build up but tends to be discrete. This is referred to as natural frequencies or principal resonance frequencies.

Parametric resonance as seen in the moving beam problem depicts a different situation where the spectrum is the union of smaller intervals. The length of these intervals is dependent on the amplitude of perturbation and approaches zero as amplitude decreases. This means that these intervals are centered about certain frequencies which are referred here as combination resonance frequencies or *quasi-frequencies*. For a classical free vibration problem as seen in the case of a simply supported beam, the frequencies evaluated from the generalized eigenvalue problem are called as natural frequencies. Ideally in the classical case, the stiffness mass and damping matrices do not change with time. Therefore, the system possesses only one set of fundamental frequencies.

However, there is a class of dynamics problems that give rise to the time-dependent boundary conditions. Vibrations of band-saw blades, robot manipulators, are some examples. The problem of a beam oscillating over two bilateral supports falls in this class. Following the definitions given by Jankovic [31], for a eigenvalue problem in which frequencies change with time there are indeed no natural frequencies associated with the system. Instead these are termed them as quasi-frequencies. In a problem involving rotating beams, Yu and Young [15] also echo Jankovic's view on time-dependent frequencies. Therefore, in this paper, all frequencies

Table 1
Beam parameters [1]

Parameters	Value
Beam stiffness (EI, N m^2)	1.0
Beam length (L, m)	1.0
Distance between supports (D, m)	0.25
Amplitude range (A, m)	0.002–0.05
Frequency (ω, rad/s)	0–50

related to the moving beam problem are referred to as quasi-frequencies or combination resonance frequencies and the same for a fixed beam are principal resonance frequencies.

6.2. Additional instability regions and Hsu's considerations

Based on the work by Hsu [32] and Buffinton and Kane [1], it is clear that the discrete A–ω values obtained by computing eigenvalues of transition matrix alone do not interpret dynamic stability in its entirety. Therefore, the A–ω pair obtained is discrete and the information that lies between, say, two values of ω is lost. In order to recover the lost information one may obtain the additional instability frequencies of the first approximation [1,3,4,32]. Referring to the beam frequencies for fixed configuration (Table 1) as λ_i and λ_j, the combination frequencies are given as

$$\lambda_c = \left|\lambda_i \pm \lambda_j\right|, \quad (i,j = 1 \cdots v). \tag{40}$$

The instability frequencies determined by Eq. (40) is valid for small values of A, i.e., $A \ll 1.0$. Further investigations by Buffinton and Kane [1] asserted the need for including more terms in Eq. (40) for λ_c, which would cover a wider spectrum of combination frequencies. These additional instability regions could be centered about a combination resonance frequency given as

$$\lambda_c = \frac{1}{m}\left|\lambda_j \pm \lambda_i\right|, \quad (i,j = 1,\ldots,v; \quad m = 2,\ldots,\infty). \tag{41}$$

When linear natural frequencies λ_i and λ_j are commensurate, internal resonance can also occur when $\lambda_c = |\lambda_i \pm (\lambda_j \pm \lambda_k)|$, $(i, j, k = 1 \ldots v)$. This even widens the instability spectrum as more combination frequencies are found. With the assumption of some form of artificial damping being present, some of these additionally found spectrums become weak and may not dominate in the instability charts. More detailed account on the parametric resonances and identification of combination frequencies can be found in Ref. [5]. The stability charts for various periodic motions of the beam are obtained and comparison of numerical results with known previous research is presented next.

7. Results and discussion

This section presents the results on the dynamic stability characteristics as the beam moves sinusoidally over supports. The analysis incorporates 5% damping, an aspect that is not considered in previous studies. The basic beam parameters given in Table 1 are same as assumed by Buffinton and Kane [1], Lee [16] and Sreeram [17]. Before presenting the instability results, the free vibration characteristics [principal resonance frequencies] of the beam with symmetric overhang are determined using the penalty-based formulation.

This is useful in the additional stability results and also as a measure of convergence for the penalty-based finite element formulation. Based on a convergence study (results not presented here), 4 elements with 3 internal node per element (4EL-3IN) is found sufficient to accurately solve the free vibration and also the dynamic stability of the moving beam. For the stability analysis, a total of 4 elements with 3 internal nodes per element are used for cases with and without damping. Table 2 shows the comparison with classical solution and the error is with in 0.3%.

Table 2
Natural frequencies of a beam with symmetric overhang

Mode number	Classical (rad/s) $D = 0.25\,\text{m}$ (a)	Penalty based (rad/s)				% Error $\frac{\lvert a-b\rvert}{a} \times 100\%$
		$D = 0.25\,\text{m}$ (b)	$D = 0.20\,\text{m}$	$D = 0.15\,\text{m}$	$D = 0.10\,\text{m}$	
1	16.246	16.246	14.762	12.934	11.356	—
2	20.771	20.776	20.054	20.607	21.655	0.024
3	117.93	117.93	102.40	85.739	73.023	—
4	136.07	136.15	132.55	135.56	139.86	0.058
5	247.47	247.81	282.78	245.37	207.66	0.137
6	386.11	386.34	357.32	382.27	392.22	0.059
7	422.58	423.69	452.45	482.48	416.84	0.262
8	702.44	703.86	632.46	654.85	677.43	0.202
9	799.47	800.41	798.61	815.20	789.32	0.117

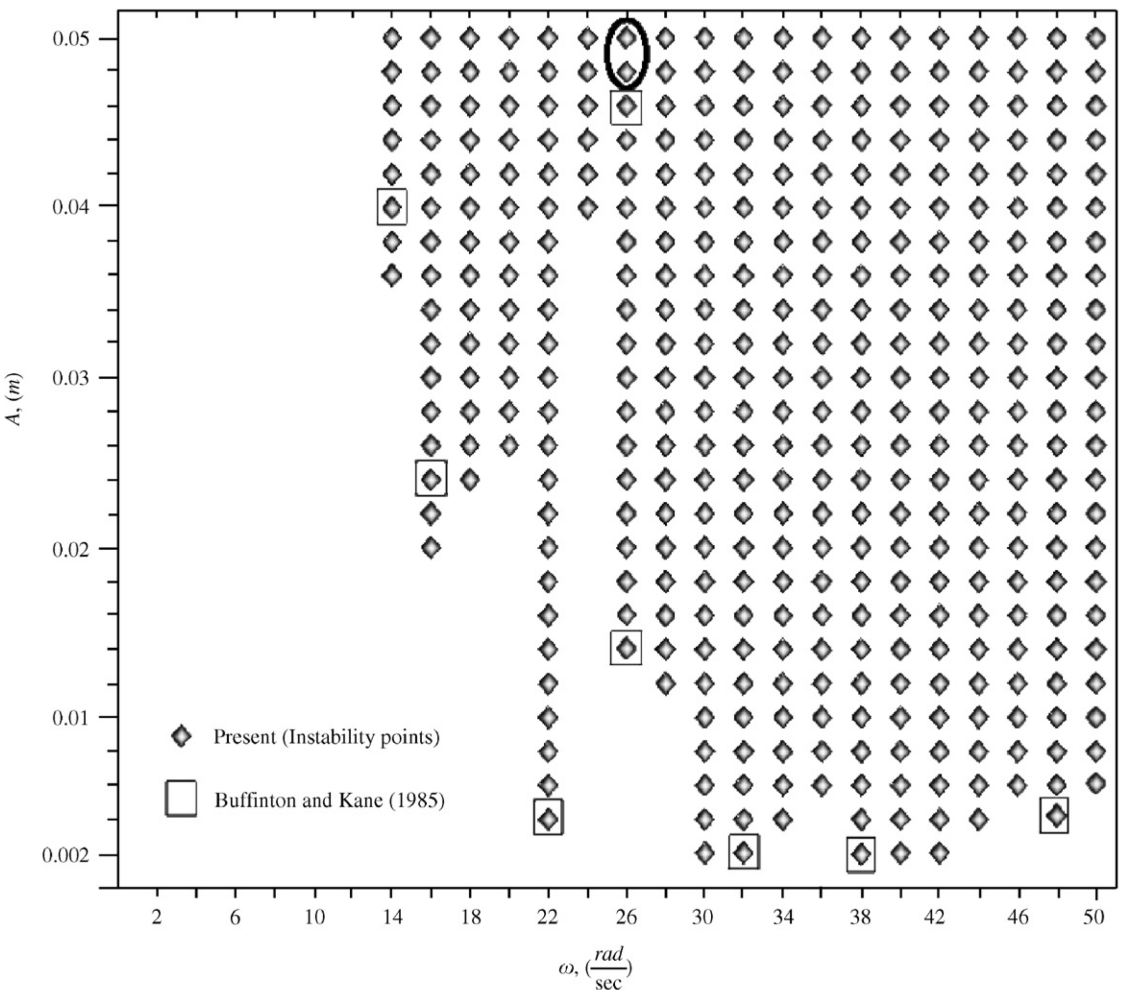

Fig. 3. Instability during periodic motion of a free–free beam over two intermediate supports $-X_F = ((L - D)/2) - A\sin(\omega t)$: $D = 0.25\,\text{m}$; no damping [4EL-3IN].

7.1. Instability predictions using Floquet theory

Having determined the principal resonance frequencies, the effects of change in amplitude of periodic motion (A), distance between supports (D) and damping on the instability is determined. The Floquet transition matrix $[\mathbf{P}_{\text{flo}}]$ is evaluated by integrating Eq. (33) between (0, T) with the initial conditions specified by the $2N \times 1$ identity matrix. Here first-order implicit methods are used to numerically integrate the $2N \times 1$ system to obtain $[\mathbf{P}_{\text{flo}}]$. The IMSL program DEVCRG is used to solve the generalized eigenproblem.

The instability regions dominate for $\omega > 22\,\text{rad/s}$ and $D = 0.25\,\text{m}$. Fig. 3 shows the case with no damping and the instability pattern compares well with that of Buffinton and Kane [1]. The region, which is highlighted in Fig. 3 by an ellipse are the additional instability points obtained in this study. In the assumed modes approach by Buffinton and Kane [1], only two modes were used to solve the stability problem. As witnessed in the present study, using more degrees of freedom (4 elements and 3 internal nodes per element), actually causes a slight shift to the left, which is clearly predicted by Buffinton and Kane [1]. Additional analysis of the response for these two A–ω pairs (0.048, 26) and (0.05, 26) is carried out and Fig. 11 reinforces the instability predictions (using Floquet theory) in Fig. 3.

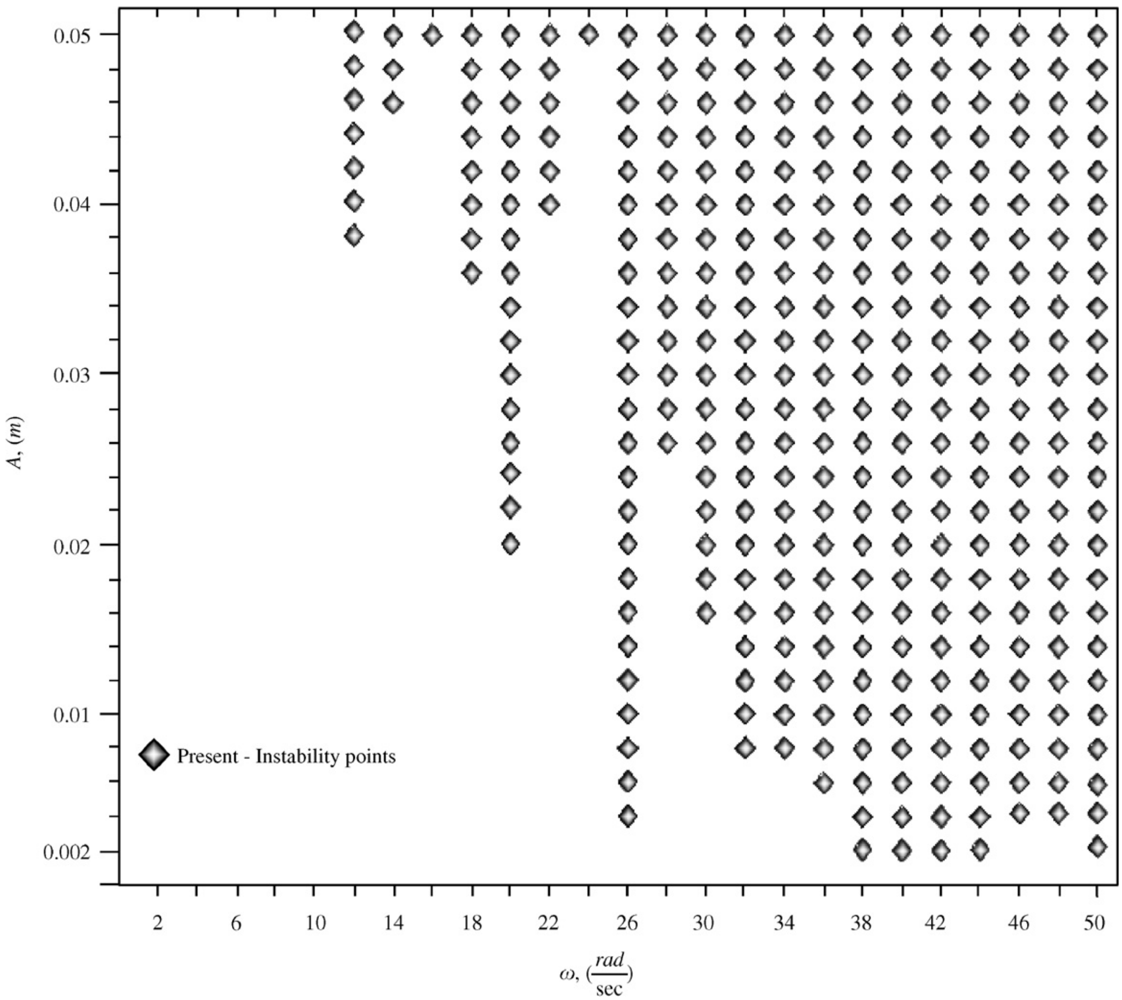

Fig. 4. Instability during periodic motion of a free–free beam over two intermediate supports $-X_F = ((L - D)/2) - A\sin(\omega t)$: $D = 0.20\,\text{m}$; no damping [4EL-3IN].

It should be noted that the Figs. 3–10 represent instability points only for discrete A–ω pairs and thus the instability information between these points are not captured. Fig. 4 shows the instability chart for $D = 0.2$ m and effects due to damping are not included. One of the main differences between Figs. 3 and 4 is that as D is decreased, it is observed that new unstable regions are uncovered towards the left of Fig. 4. These regions were not identified in Fig. 3. Subsequent results show the effect of further decreasing the distance between the supports on the stability characteristics. For $D = 0.1$ m, the instability pattern in the frequency range 12 rad/s (Fig. 6) is distinct compared to $D = 0.25$ m. In general, for all cases with $D < 0.25$ m more instability points are found towards the lower frequencies, whereas the pattern is similar between higher frequencies in the range 20–50 rad/s.

It is observed that for any decrease in the value of D has an adverse effect on the stability. More instability regions are uncovered as D is decreased. Some of the weaker instability spectrum are uncovered (by Hsu's first and higher approximations) since decreasing D results in the larger span of the beam unsupported. Interestingly, consistent with Hsu's predictions in Table 3, Figs. 3–6 indicate that some of the instability regions are seen around the frequencies of first approximation. Since the A–ω plot is discrete, only some of the instability regions are confirmed using the Floquet theory. For example, $\max(\lambda_1 + \lambda_2)$ in Tables 3 and 4

$$\min\left[\frac{(\lambda_1 + \lambda_2)}{k}, k = 2\right]$$

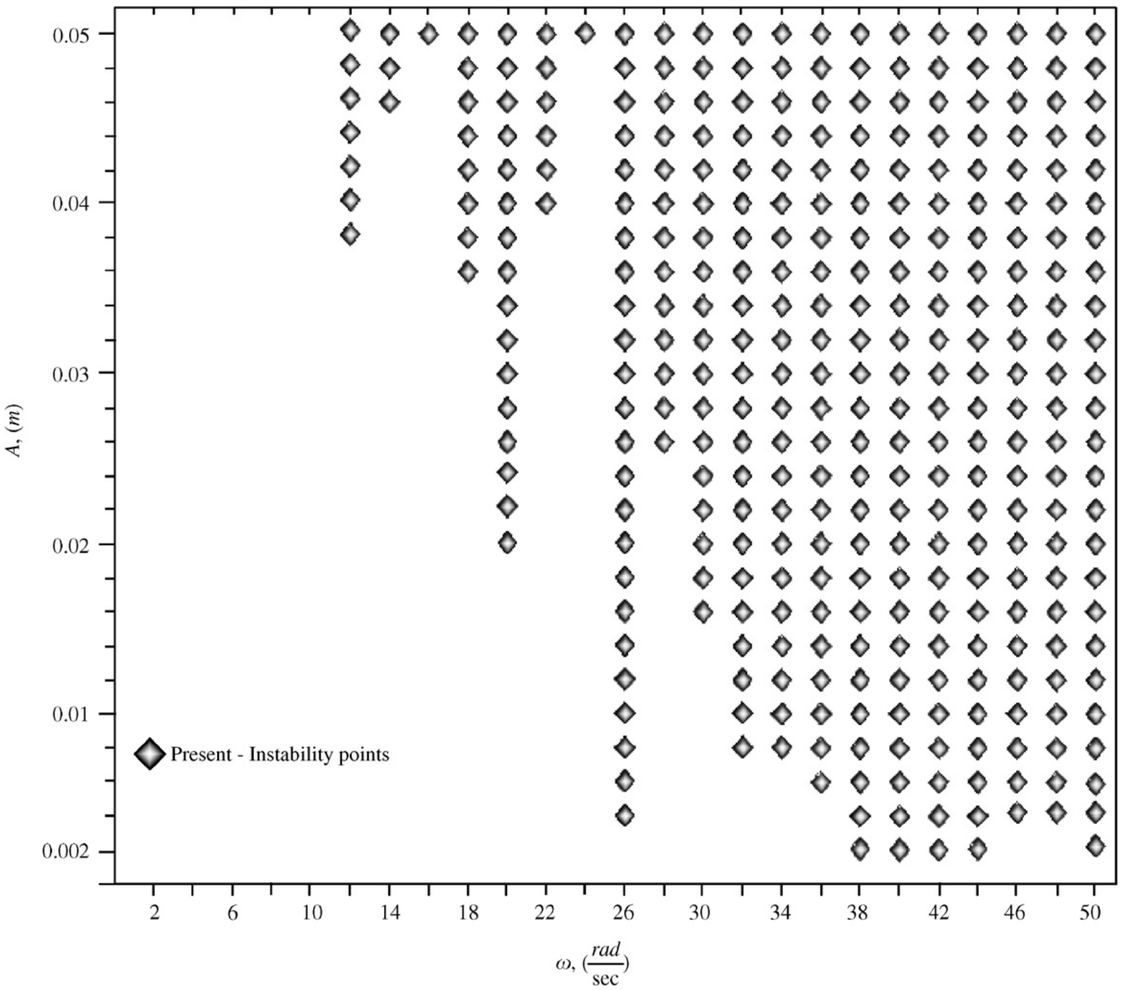

Fig. 5. Instability during periodic motion of a free–free beam over two intermediate supports $-X_F = ((L - D)/2) - A\sin(\omega t)$: $D = 0.15$ m; no damping [4EL-3IN].

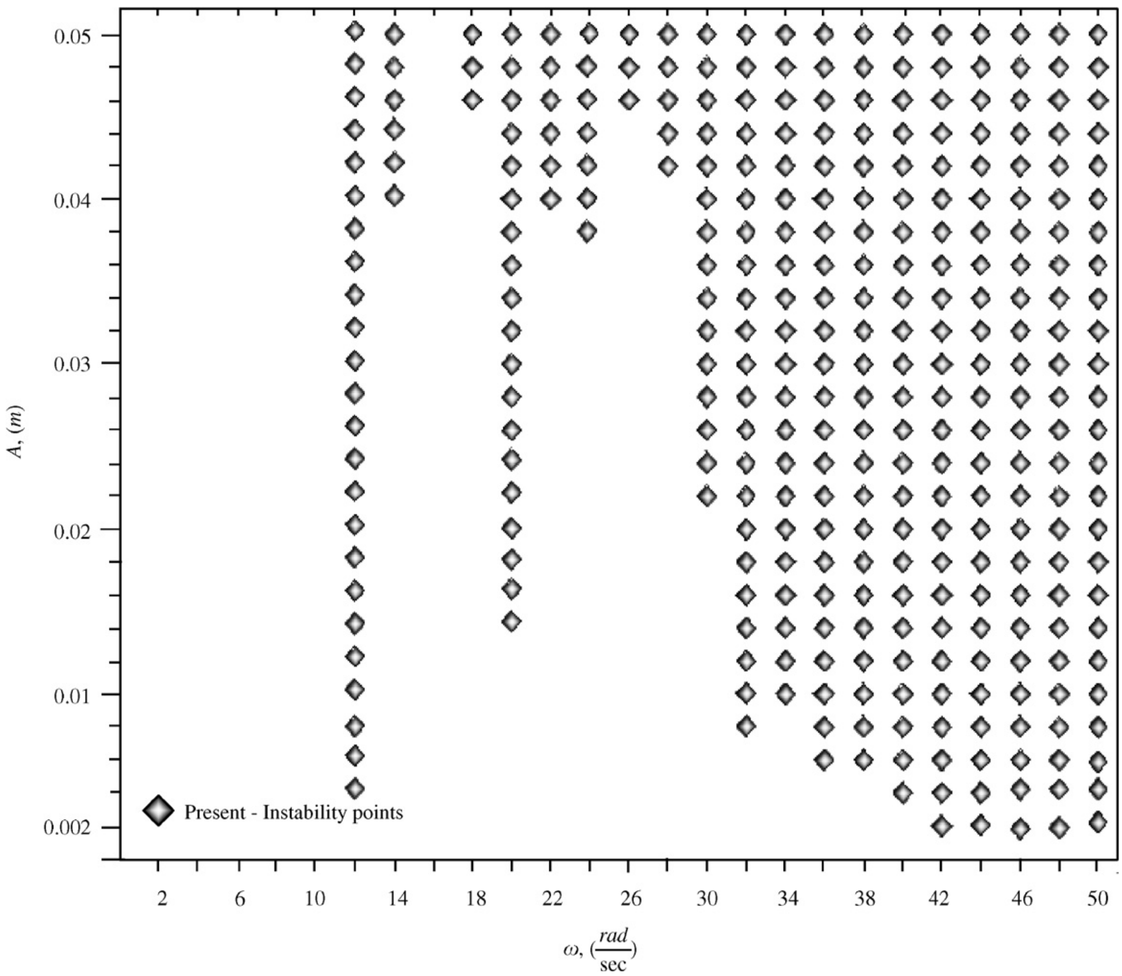

Fig. 6. Instability during periodic motion of a free–free beam over two intermediate supports $-X_F = ((L-D)/2) - A\sin(\omega t)$: $D = 0.1$ m; no damping [4EL-3IN].

shows that the instability frequencies range between 16.5 and 37.022 rad/s. Several discrete frequencies causing instability in this range are confirmed in the predictions using Floquet's theory as shown in Figs. 3–6. Table 4 also shows additional instability frequencies given by $|\lambda_1 - \lambda_2|/k$ in the lower range, however, these regions could not be identified using Floquet theory.

For all instability charts (Figs. 3–6) without damping the region towards higher frequencies do not change significantly. For cases with damping included, the instability at lower frequency range disappears whereas the influence of damping in the higher frequency range is minimal. For the damped case in Fig. 7, the presence of damping seem to eliminate some of the weaker instability regions, particularly in the lower frequency range, $\omega = 1$–14 rad/s. However, the stronger instability regions in the vicinity of the first two principal resonance frequencies of the stationary beam remain predominantly unchanged and can be seen from Figs. 7–10.

7.2. Identification of additional instability regions

Based on Hsu's approximations, the additional combination resonance frequencies are obtained for a range of D and are shown in Table 3. The table lists the frequencies of the first approximations. For the cases with decreasing values of D and damping inclusive, no known attempts have studied this in the

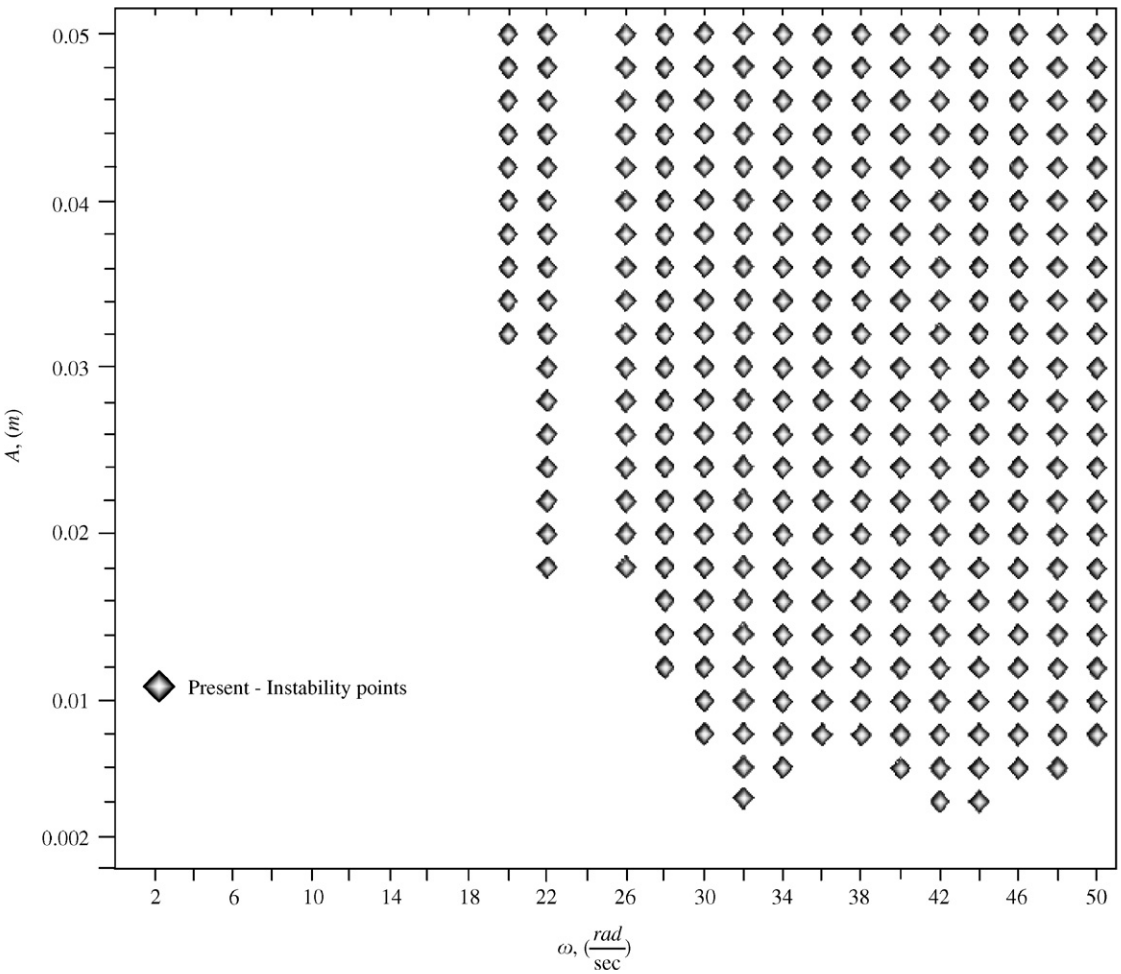

Fig. 7. Instability during periodic motion of a free–free beam over two intermediate supports $-X_F = ((L-D)/2) - A\sin(\omega t)$: $D = 0.25$ m; 5% damping [4EL-3IN].

context of a beam moving over supports and hence this work identifies the effect of various factors that determine instability.

For the union of the first two fundamental frequencies, the instability frequency in all the cases is centered about 33 and 37 rad/s. It should be noted that primarily the instability regions are in the neighborhood of the first few frequencies, i.e., modes 1 and 2. This is evident from Figs. 3–10 that instability regions are witnessed in the vicinity of these fundamental frequencies. For $D = 0.25$ m, the instability is observed as far as $A = 0.05$ m starting from a lower amplitude of 0.004 m, though the stability charts do not capture this region. This is expected since the second fundamental frequency is close to 21 rad/s (Table 2). For the case when $D = 0.15$ m (Fig. 5), a combination frequency appear to center around 34 rad/s. This is consistent with the predictions in Table 3. Depending on the type of nonlinearity in variation, the frequencies can be identified in the vicinity of a value of combination frequency. For lower values of D, more unstable frequencies are detected to the left region in Figs. 4–6. This is due to the lowering magnitude of the first fundamental frequency. However, some of these initially dominant instability regions are seen to fade away with the inclusion of damping. Hsu [32] as well as Buffinton and Kane [1] argued that the combination frequencies obtained using $k = 1$, 2 are the most significant. The instability chart results for $D = 0.25$ m agree qualitatively with that of Buffinton and Kane [1], with some additional instability frequencies newly identified.

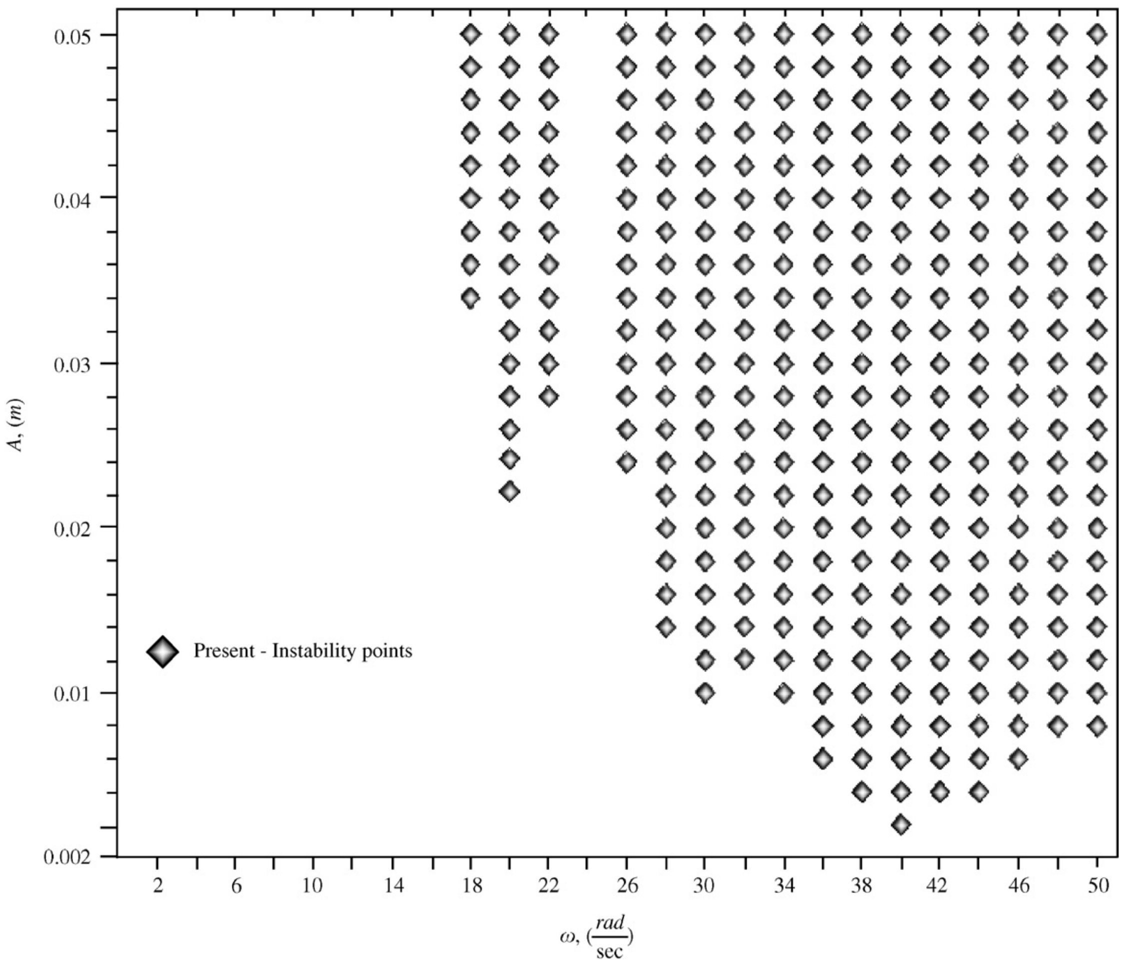

Fig. 8. Instability during periodic motion of a free–free beam over two intermediate supports $-X_F = ((L-D)/2) - A\sin(\omega t)$: $D = 0.20\,\text{m}$; 5% damping [4EL-3IN].

7.3. Analysis of results

Floquet–Lyapunov theory has been used to predict the instability regions as shown by charts (Figs. 3–10). These instability regions are discrete for A–ω pairs and as predicted by Buffinton and Kane [1]. Additional instability regions do exist and are determined by Hsu's approximations. The Floquet–Lyapunov theory is a specialized procedure particularly suited for large amplitude problems as in the present case. As seen from the stability charts, the instability is predicted based on a combination of parameters such as D, A and ω. However, the additional instability regions obtained using Hsu's approximations are based on a combination of first few frequencies of a symmetrically overhang beam. The predictions based on Hsu's approximations do not take into consideration the factors such as application of a driving force on a specific point on the beam [22]. The effect of driving force point on the stability characteristics is covered in Ref. [22] and is not addressed here. The effect of driving force point on the response has been evaluated in Ref. [19] for repositional motions (as in forward and reverse repositional motions). This suggests the instability determinations using Floquet's theory as well as those using the Hsu's approximations are complementary to each other. Based on the Hsu's determinations, Floquet–Lypunov theory can be applied to determine if any specific A–ω pairs would result in instability.

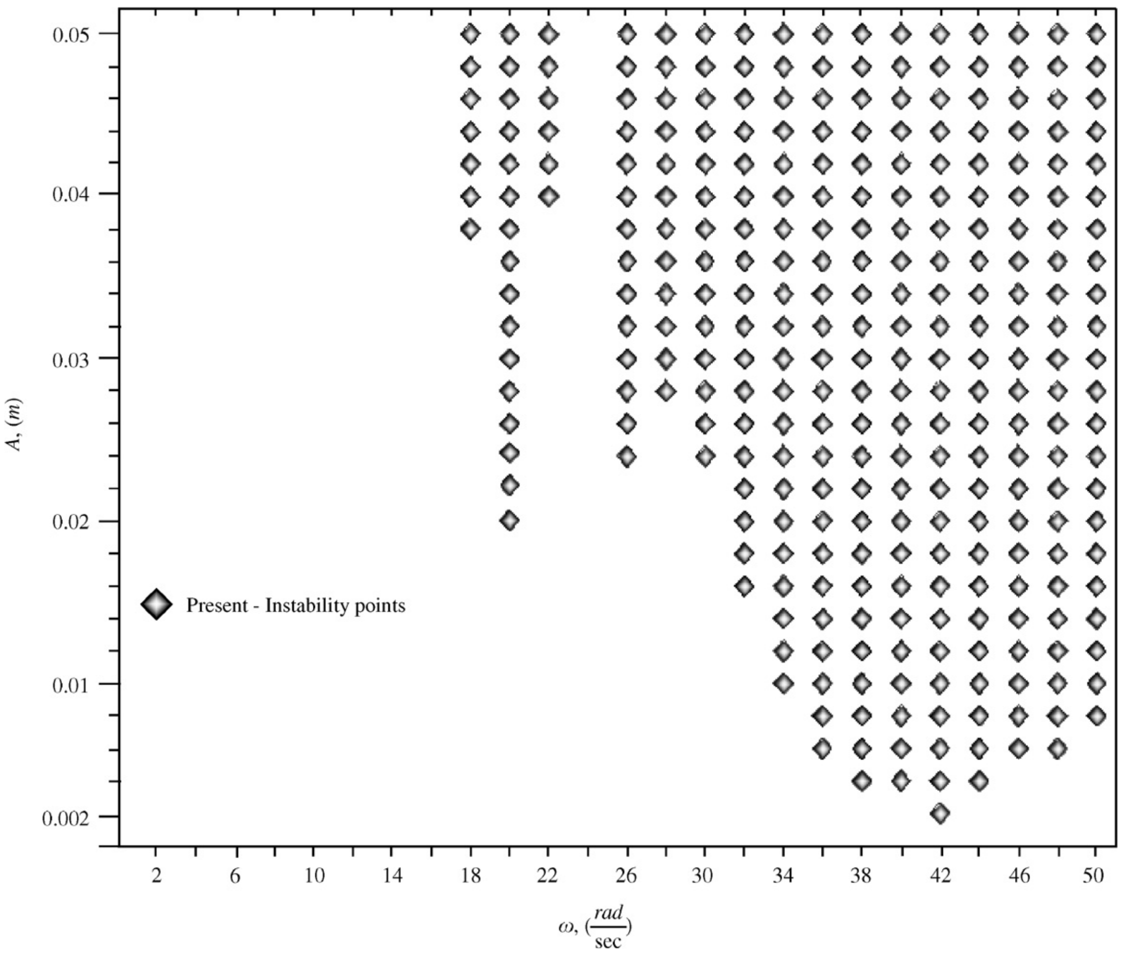

Fig. 9. Instability during periodic motion of a free–free beam over two intermediate supports $-X_F = ((L-D)/2) - A\sin(\omega t)$: $D = 0.15\,\text{m}$; 5% damping [4EL-3IN].

8. Conclusions

A numerical formulation is presented here to address the dynamic stability of a beam moving sinusoidally over supports. Two key aspects in this unique problem is the changing nature of the support locations and necessity to enforce the essential conditions during the periodic motion, and reduction of dynamic equations in the first-order state-space form. With the aim to over come the drawbacks of Lagrangian multipliers, the penalty function method is adopted to enforce the time-dependent essential conditions. The penalty formulation also results in a positive definite system. Floquet–Lyapunov theory is used to investigate the instability during the periodic motion of the beam. Wherever possible, a comparison is drawn with known previous research and for the stability case with $D = 0.25\,\text{m}$ and no damping included the instability patterns compare well with that of Buffinton and Kane [1]. Other results with $D < 0.25\,\text{m}$ inclusive of damping have not been reported by previous authors for the moving beam problem. This work successfully applied the penalty-based formulation for a unique class of dynamics problems. The formulation presented here is less complex compared to techniques based on assumed modes and numerically more approachable than Lagrangian multipliers. Yet, other options for enforcing time-dependent essential conditions remain to be explored. One possibility is to develop a finite element-based scheme where the element length and internal nodes are generated dynamically such that at any given time the nodes are present at support locations to enforce the

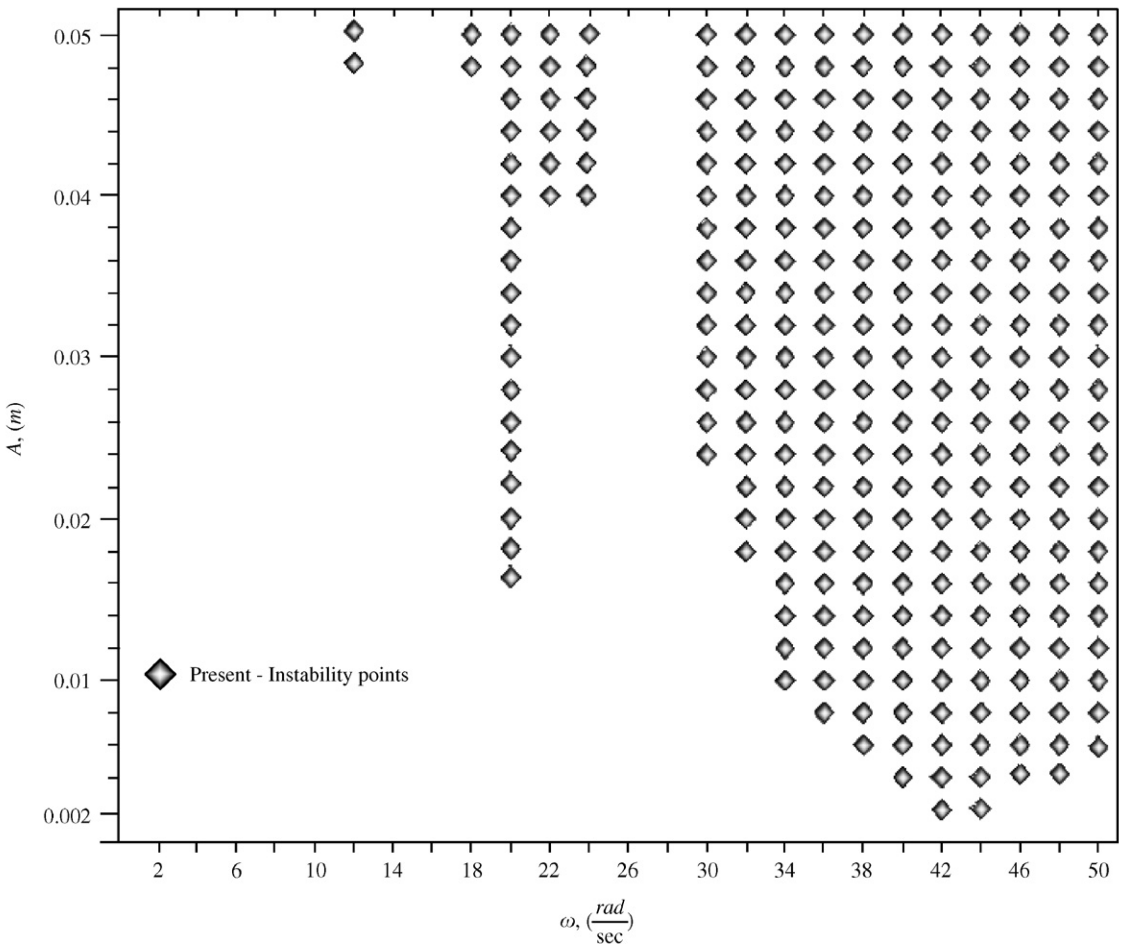

Fig. 10. Instability during periodic motion of a free–free beam over two intermediate supports $-X_F = ((L-D)/2) - A\sin(\omega t)$: $D = 0.10\,\mathrm{m}$; 5% damping [4EL-3IN].

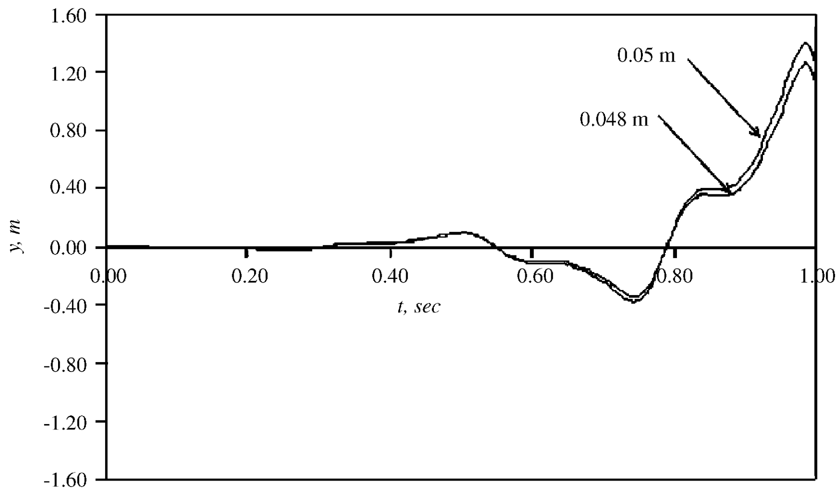

Fig. 11. Instability during periodic motion of a free–free beam over two intermediate supports $-\omega = 26\,\mathrm{rad/s}$ and $A = 0.048\,\mathrm{m}$, 0.05 m.

Table 3
Combination resonance frequencies of first approximation

Distance between the supports (D, m)	Principal resonance frequencies (rad/s)		Combination resonance frequencies λ_c, (rad/s)	
	λ_1	λ_2	$\lambda_1+\lambda_2$	$\|\lambda_1-\lambda_2\|$
0.25	16.246	20.776	37.022	4.530
0.20	14.762	20.054	34.817	5.291
0.15	12.934	20.606	33.540	7.672
0.10	11.356	21.655	33.011	10.29

Table 4
Additional instability frequencies of higher approximations

Distance between supports (D, m)	Principal resonance frequencies (rad/s)		Combination resonance frequencies based on Hsu's approximations, λ_c (rad/s)					
			$\frac{\lambda_1+\lambda_2}{k}$			$\frac{\|\lambda_1-\lambda_2\|}{k}$		
	λ_1	λ_2	$k=2$	$k=3$	$k=4$	$k=2$	$k=3$	$k=4$
0.25	16.246	20.776	18.511	12.340	9.255	2.265	1.510	1.132
0.20	14.762	20.054	17.408	11.605	8.704	2.645	1.763	1.322
0.15	12.934	20.606	16.770	11.180	8.385	3.836	2.557	1.918
0.10	11.356	21.655	16.505	11.003	8.252	5.149	3.433	2.574

essential conditions. Such schemes may be able to offer computational advantages over mixed formulations especially when the number of supports is more. The feasibility of such a scheme in the context of a unique class of problems in dynamics is an issue that needs to be explored in the future.

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